San Diego Math League High School Division, Round 2a
January 29, 2011

1. The value of $\dfrac{2.011 \times 2011}{20.11 \times 201.1}$ is

   (A) 0.01 (B) 0.1 (C) 1 (D) 10 (E) 100

2. The tens digit of the product $1 \times 2 \times 3 \times \cdots \times 98 \times 99$ is

   (A) 0 (B) 1 (C) 2 (D) 4 (E) 9

3. A 45-gallon cask is filled with 45 gallons of wine. Nine gallons are removed, and the cask is refilled with water. Then nine gallons of the mixture are removed, and the cask is refilled with water again. The ratio of water to wine in the final mixture is

   (A) $6:25$ (B) $9:16$ (C) $2:3$ (D) $2:5$ (E) none of these

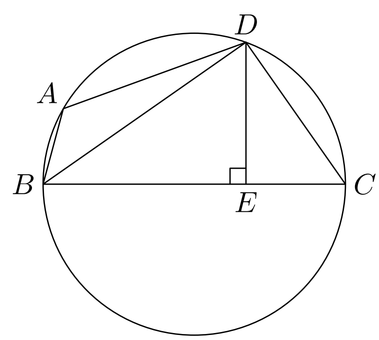
4. How many integers between 100 and 1000 have the sum of its digits equal to 10?

   (A) 36 (B) 54 (C) 55 (D) 62 (E) 63

5. Let r and s be the roots of the equation $x^2 - 5x - 3 = 0$. If $r + s^{-1}$ and $s + r^{-1}$ are the roots of the equation $x^2 + px - 2q = 0$, what is q?

   (A) 1 (B) $\frac{2}{3}$ (C) -3 (D) $-\frac{4}{3}$ (E) $\frac{4}{3}$

6. If BC is a diameter of the circle shown, then the sine of angle BDE equals the sine of angle



   (A) EDC (B) ADB (C) BAD (D) ABD (E) DBC

7. Let x be a real number that satisfies the equation $16(\log_9 x)^4 = (\log_3 x^3)^2 + 10$. Determine $(\log_9 x)^2$.

   (A) 10 (B) 4 (C) $\sqrt{10}$ (D) $\frac{5}{2}$ (E) $\frac{\sqrt{5}}{2}$

8. By a *binary word of length* n, we mean a string of n digits each of which is either 0 or 1. By the *distance* between two binary words, we mean the number of digits where the two words differ. For example, the distance between the two words 100010 and 010011 is 3. How many binary words of length 6 have distance at most 4 from the the word 100010?

   (A) 6 (B) 18 (C) 24 (D) 31 (E) 57

9. The multiples of 2 and 5 are removed from the set of positive integers $\{1, 2, 3, \ldots, 10n\}$, where n is a positive integer. The sum of the remaining integers is

   (A) $10n^3 - 40n^2 + 110n - 60$ (B) $30n^2 - 30n + 20$ (C) $15n^2 + 5n$ (D) $20n^2$

   (E) none of these

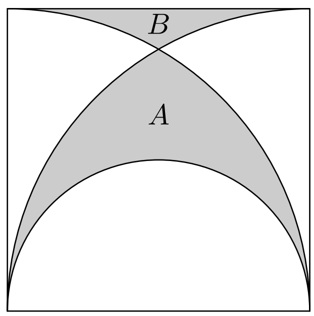
10. Let k be the smallest positive integer such that the numbers $1+k$, $1+2k$, and $1+8k$ are divisible by 3, 5, and 7, respectively. Then k lies in the interval

    (A) 0 to 20 (B) 21 to 40 (C) 41 to 60 (D) 61 to 80 (E) 81 to 100

11. For an integer n, let $T_n = \frac{1}{2}n(n+1)$. Find the number of ordered pairs of integers (m, n) such that $T_m - T_n = 2011$.

    (A) 1 (B) 2 (C) 4 (D) 8 (E) 16

12. The diagram shows a semicircle and two quarter circles inscribed in a square of side length 2. The difference between the area of the shaded region A and the area of the shaded region B equals



    (A) $\frac{3}{2}\pi - 4$ (B) $\frac{1}{3}\pi - \frac{1}{3}$ (C) $\frac{3}{2} - \frac{1}{4}\pi$ (D) $4 - \pi$ (E) $\frac{1}{4}\pi$

13. An ant wishes to travel from one vertex of a wooden block with dimensions $2 \times 4 \times 8$ to the opposite vertex (the farthest vertex away). The shortest distance it can walk is

    (A) 14 (B) $2 + \sqrt{80}$ (C) $8 + \sqrt{20}$ (D) $4 + \sqrt{68}$ (E) 10

14. Fifteen lines are drawn in a plane, with no three concurrent and no two parallel, dividing the plane into 121 non-overlapping regions. Some of these regions are completely bounded by line segments and some are not. The number that are completely bounded by line segments is

    (A) 45 (B) 81 (C) 30 (D) 75 (E) 91

15. Ophelia, Ada, Max, and Kapila play a game, where each person writes down a random positive integer on a separate slip of paper. For each player, the probability of writing down the positive integer n is 2^{-n}, and these probabilities are independent.

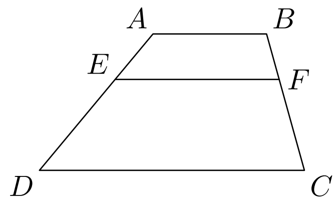
    The person who writes down the smallest positive integer *that no one else has also written down* is the winner. For example, if the four positive integers are 2, 7, 10, 23, then the person who wrote 2 wins, but if the four positive integers are 3, 3, 8, 12, then the person who wrote 8 wins.

    Note that there is a winner only if there is a unique smallest positive integer, so it is possible that the game has no winner. For example, if the four positive integers are 5, 5, 9, 9, then there is no winner.

    Find the probability that Ophelia is the winner.

    (A) $\frac{1}{4}$ (B) $\frac{1}{5}$ (C) $\frac{11}{60}$ (D) $\frac{8}{45}$ (E) $\frac{43}{180}$

1. In the diagram, AB, CD, and EF are parallel, with $AB = 3.6$, $CD = 13.2$, $AE = 2.2$, and $DE = 6.6$. Find the length of EF.



2. In how many ways can the letters of the word OLYMPIAD be arranged if the vowels must be in alphabetical order? (In this problem, the letter Y should always be considered a vowel.)

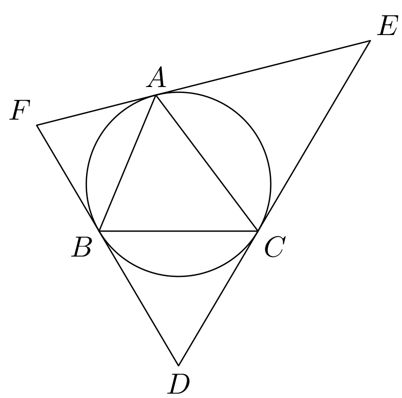
3. X and Y are positive integers. The sum of the digits of X is 53, and the sum of the digits of Y is 47. If the addition of X and Y involves exactly 5 carries, find the sum of the digits of $X + Y$. (For example, when we add 234 and 185, there is one carry from adding the digits 3 and 8.)

4. The first, fourth, and eighth terms of a nonconstant arithmetic sequence form a geometric sequence. If its twentieth term is 56, what is its tenth term?

5. If m and n are positive integers with $m \leq 2011$, and $r = 2 - \frac{m}{n} > 0$, find the smallest possible value of r.

6. Let $x = \sqrt[3]{2} + \sqrt[3]{4}$. There exist unique integers a, b, and c such that $x^5 = ax^2 + bx + c$. Find $a + b + c$.

7. Given acute triangle ABC, the tangents to the circumcircle at vertices A, B, and C are drawn. These tangents determine triangle DEF, as shown below.



If $\cos A = \frac{3}{5}$, $\cos B = \frac{1}{\sqrt{10}}$, and $BC = \frac{24}{5}$, then find the area of triangle DEF.

8. Find the number of ways of packing a $2 \times 2 \times 4$ box with eight $1 \times 1 \times 2$ bricks.